Assignment 5

Deadline: Feb 22, 2019

Hand in: Supp. Ex no 2,3, 5, and 6.

Section 7.1: No. 1, 2, 8, 11, 14, 15. (In 14(d) you apply Theorem 2.6 instead of Example 7.1.4.),

Section 7.2: No. 18, 19.

Supplementary Exercise

1. Let P be the partition $\{-1, -\frac{1}{2}, 0, \frac{1}{3}, 1\}$ of $[-1,1]$. Define $f : [-1,1]$ by
$$f(x) = \begin{cases} -x & \text{if } x \in [-1,0], \\ -x+1 & \text{if } x \in (0,1]. \end{cases}$$
   (a) Find the Darboux upper and lower sums for f. Explain why the Darboux upper sum is not a Riemann sum.
   (b) Use the integrability criterion to show that f is integrable and find its integral.
2. Prove Cauchy criterion for integrability: f is integrable on $[a,b]$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any two tagged partitions $\dot{P}, \dot{Q}$ with length less than δ,
$$|S(f,\dot{P}) - S(f,\dot{Q})| < \varepsilon,$$
holds. (This criterion is proved in the text; pretend that it is not there.)
3. Let $f_+(x) = \max\{f(x), 0\}$ and $f_-(x) = -\min\{f(x), 0\}$. Show that f_+ and f_- are both integrable when f is integrable on $[a,b]$.
4. Let g be differed from f by finitely many points. Show that g is integrable if f is integrable over $[a,b]$ and they have the same integral over $[a,b]$.
5. Let f be non-negative and continuous on $[a,b]$. Show that $\int_a^b f = 0$ if and only if $f \equiv 0$.
6. Let $f \in \mathcal{R}[a,b]$ and $g \in C^1[c,d]$ where $f[a,b] \subset [c,d]$. Show that the composite $g \circ f \in \mathcal{R}[a,b]$. Here C^1 means continuously differentiable.
7. (Optional). Let $f \in \mathcal{R}[a,b]$ and $g \in C[c,d]$ where $f[a,b] \subset [c,d]$. Show that the composite $g \circ f \in \mathcal{R}[a,b]$. Hint: For $\varepsilon > 0$, fix δ_0 such that $|g(z_1) - g(z_2)| < \varepsilon$ for $|z_1 - z_2| < \delta_0$. For ε, $\delta_0 > 0$, there exists a partition P such that $\sum_j osc_{I_j} f \Delta x_j < \varepsilon\delta_0$. Then apply the Second Criterion.
8. Let f be a continuous function on $[a,b]$ and g a nonnegative integrable function on the same interval. Prove the mean-value theorem for integral:
$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx,$$
for some $c \in [a,b]$.